OPMT 5701: Calculus for Utility Problems

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1 Using Calculus For Utility Maximization Problems

1.1 Review of Some Derivative Rules

1. Partial Derivative Rules:

$U = xy$	$\partial U/\partial x = y$	$\partial U/\partial y = x$
$U = x^a y^b$	$\partial U/\partial x = ax^{a-1}y^b$	$\partial U/\partial y = bx^a y^{b-1}$
$U = x^a y^{-b} = \frac{x^a}{y^b}$	$\partial U/\partial x = ax^{a-1}y^{-b}$	$\partial U/\partial y = -bx^a y^{-b-1}$
$U = ax + by$	$\partial U/\partial x = a$	$\partial U/\partial y = b$
$U = ax^{1/2} + by^{1/2}$	$\partial U/\partial x = a\left(\frac{1}{2}\right)x^{-1/2}$	$\partial U/\partial y = b\left(\frac{1}{2}\right)y^{-1/2}$

2. Logarithm (Natural log) $\ln x$

   (a) Rules of natural log

If	$Then$
$y = AB$	$\ln y = \ln(AB) = \ln A + \ln B$
$y = A/B$	$\ln y = \ln A - \ln B$
$y = A^b$	$\ln y = \ln(A^b) = b\ln A$

   NOTE: $\ln(A + B) \neq \ln A + \ln B$

   (b) derivatives

IF	$THEN$
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \ln(f(x))$	$\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

   (c) Examples

If	$Then$
$y = \ln(x^2 - 2x)$	$dy/dx = \frac{1}{(x^2-2x)}(2x - 2)$
$y = \ln(x^{1/2}) = \frac{1}{2}\ln x$	$dy/dx = \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) = \frac{1}{2x}$

3. The Number e

$$\begin{aligned} \text{if } y &= e^x \text{ then } \frac{dy}{dx} = e^x \\ \text{if } y &= e^{f(x)} \text{ then } \frac{dy}{dx} = e^{f(x)} \cdot f'(x) \end{aligned}$$

   (a) Examples

$$\begin{array}{ll} y = e^{3x} & \frac{dy}{dx} = e^{3x}(3) \\ y = e^{7x^3} & \frac{dy}{dx} = e^{7x^3}(21x^2) \\ y = e^{rt} & \frac{dy}{dt} = re^{rt} \end{array}$$

1.2 Finding the MRS from Utility functions

EXAMPLE: Find the total differential for the following utility functions

1. $U(x_1, x_2) = ax_1 + bx_2$ where $(a, b > 0)$
2. $U(x_1, x_2) = x_1^2 + x_2^3 + x_1x_2$
3. $U(x_1, x_2) = x_1^a x_2^b$ where $(a, b > 0)$
4. $U(x_1, x_2) = \alpha \ln c_1 + \beta \ln c_2$ where $(\alpha, \beta > 0)$

Answers:
1. $\frac{\partial U}{\partial x_1} = U_1 = a \quad \frac{\partial U}{\partial x_2} = U_2 = b$
and

$$dU = U_1 dx_1 + U_2 dx_2 = adx_1 + bdx_2 = 0$$

If we rearrange to get dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = -\frac{U_1}{U_2} = -\frac{a}{b}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{a}{b}$$

2. $\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2 \quad \frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1$
and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + (3x_2^2 + x_1)dx_2 = 0$$

Find dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

iii) $\frac{\partial U}{\partial x_1} = U_1 = ax_1^{a-1}x_2^b$ $\qquad$ $\frac{\partial U}{\partial x_2} = U_2 = bx_1^a x_2^{b-1}$
and

$$dU = \left(ax_1^{a-1}x_2^b\right)dx_1 + \left(bx_1^a x_2^{b-1}\right)dx_2 = 0$$

Rearrange to get

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{ax_1^{a-1}x_2^b}{bx_1^a x_2^{b-1}} = -\frac{ax_2}{bx_1}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{ax_2}{bx_1}$$

iv) $\frac{\partial U}{\partial c_1} = U_1 = \alpha\left(\frac{1}{c_1}\right)dc_1 = \left(\frac{\alpha}{c_1}\right)dc_1$ $\qquad$ $\frac{\partial U}{\partial x_2} = U_2 = \beta\left(\frac{1}{c_2}\right)dc_2 = \left(\frac{\beta}{c_2}\right)dc_2$
and

$$dU = \left(\frac{\alpha}{c_1}\right)dc_1 + \left(\frac{\beta}{c_2}\right)dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{U_1}{U_2} = \frac{\left(\frac{\alpha}{c_1}\right)}{\left(\frac{\beta}{c_2}\right)} = -\frac{\alpha c_2}{\beta c_1}$$

The MRS is the Absolute value of $\frac{dc_2}{dc_1}$:

$$\begin{aligned} MRS &= \frac{\alpha c_2}{\beta c_1} = (1+r) \\ c_2 &= \beta c_1(1+r) \qquad \textit{and} \qquad c_1 = \frac{c_2}{\beta(1+r)} \end{aligned}$$

1.3 Application: Intertemporal Utility Maximization

Consider a simple two period model where a consumer's utility is a function of consumption in both periods. Let the consumer's utility function be

$$U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

where c_1 is consumption in period one and c_2 is consumption in period two. The consumer is also endowments of y_1 in period one and y_2 in period two.

Let r denote a market interest rate with the consumer can choose to borrow or lend across the two periods. The consumer's intertemporal budget constraint is

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

1.3.1 Method One:Find MRS and Substitute

Differentiate the Utility function

$$dU = \left(\frac{1}{c_1}\right) dc_1 + \left(\frac{\beta}{c_2}\right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{c_2}{\beta c_1}$$

The MRS is the Absolute value of $\frac{dc_2}{dc_1}$:

$$MRS = \frac{c_2}{\beta c_1}$$

substitute into the budget constraint

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= c_1 + \frac{\beta c_1(1+r)}{1+r} = (1+\beta)c_1 \\ c_1^* &= \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)} \end{aligned}$$

Similarly, solving for c_2^* using the first order conditions

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r} \\ (1+r)y_1 + y_2 &= \left(\frac{1}{\beta} + 1\right) c_2 \\ c_2^* &= \frac{(1+r)y_1 + y_2}{\frac{1}{\beta} + 1} \end{aligned}$$

1.3.2 Method Two: Use the Lagrange Multiplier Method

The Lagrangian for this utility maximization problem is

$$L = \ln c_1 + \beta \ln c_2 + \lambda \left(y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right)$$

The first order conditions are

$$\begin{aligned} \frac{\partial L}{\partial \lambda} &= y_1 + \tfrac{y_2}{1+r} - c_1 - \tfrac{c_2}{1+r} = 0 \\ \frac{\partial L}{\partial C_1} &= \tfrac{1}{c_1} - \lambda = 0 \\ \frac{\partial L}{\partial C_1} &= \tfrac{\beta}{c_2} - \tfrac{\lambda}{1+r} = 0 \end{aligned}$$

Combining the last two first order equations to eliminate λ gives us

$$\begin{aligned} \frac{1/c_1}{\beta/c_2} &= \frac{c_2}{\beta c_1} = \frac{\lambda}{\frac{\lambda}{1+r}} = 1 + r \\ c_2 &= \beta c_1 (1+r) \qquad \textit{and} \qquad c_1 = \frac{c_2}{\beta(1+r)} \end{aligned}$$

sub into the Budget constraint

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= c_1 + \frac{\beta c_1 (1+r)}{1+r} = (1+\beta) c_1 \\ c_1^* &= \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)} \end{aligned}$$

Similarly, solving for c_2^* using the first order conditions

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r} \\ (1+r) y_1 + y_2 &= \left(\frac{1}{\beta} + 1 \right) c_2 \\ c_2^* &= \frac{(1+r) y_1 + y_2}{\frac{1}{\beta} + 1} \end{aligned}$$