OPMT 5701
Multivariable Calculus

Partial Derivatives

Single variable calculus is really just a "special case" of multivariable calculus. For the function $y = f(x)$, we assumed that y was the endogenous variable, x was the exogenous variable and everything else was a parameter. For example, given the equations

$$y = a + bx$$

or

$$y = ax^n$$

we automatically treated a, b, and n as constants and took the derivative of y with respect to x (dy/dx). However, what if we decided to treat x as a constant and take the derivative with respect to one of the other variables? Nothing precludes us from doing this. Consider the equation

$$y = ax$$

where

$$\frac{dy}{dx} = a$$

Now suppose we find the derivative of y with respect to a, *but TREAT x as the constant.* Then

$$\frac{dy}{da} = x$$

Here we just "reversed" the roles played by a and x in our equation.

Two Variable Case:

let $z = f(x, y)$, which means **"z is a function of x and y"**. In this case z is the endogenous (dependent) variable and both x and y are the exogenous (independent) variables. To measure the the effect of a change in a single independent variable (x or y) on the dependent variable (z) we use what is known as the *PARTIAL DERIVATIVE.* The partial derivative of z with respect to x measures the instantaneous change in the function as x changes while *HOLDING y constant.* Similarly, we would hold x constant if we wanted to evaluate the effect of a change in y on z. Formally:

- $\frac{\partial z}{\partial x}$ is the **"partial derivative"** of z with respect to x, treating y as a constant. Sometimes written as f_x.
- $\frac{\partial z}{\partial y}$ is the **"partial derivative"** of z with respect to y, treating x as a constant. Sometimes written as f_y.

The "∂" symbol ("bent over" lower case D) is called the "partial" symbol. It is interpreted in exactly the same way as $\frac{dy}{dx}$ from single variable calculus. The ∂ symbol simply serves to remind us that there are other variables in the equation, but for the purposes of the current exercise, these other variables are held constant.

EXAMPLES:

$$\begin{array}{lll} z = x + y & \partial z/\partial x = 1 & \partial z/\partial y = 1 \\ z = xy & \partial z/\partial x = y & \partial z/\partial y = x \\ z = x^2y^2 & \partial z/\partial x = 2(y^2)x & \partial z/\partial y = 2(x^2)y \\ z = x^2y^3 + 2x + 4y & \partial z/\partial x = 2xy^3 + 2 & \partial z/\partial y = 3x^2y^2 + 4 \end{array}$$

- **REMEMBER:** When you are taking a partial derivative you treat the other variables in the equation as constants!

Rules of Partial Differentiation

Product Rule: given $z = g(x,y) \cdot h(x,y)$

$$\frac{\partial z}{\partial x} = g(x,y) \cdot \frac{\partial h}{\partial x} + h(x,y) \cdot \frac{\partial g}{\partial x}$$
$$\frac{\partial z}{\partial y} = g(x,y) \cdot \frac{\partial h}{\partial y} + h(x,y) \cdot \frac{\partial g}{\partial y}$$

Quotient Rule: given $z = \frac{g(x,y)}{h(x,y)}$ and $h(x,y) \neq 0$

$$\frac{\partial z}{\partial x} = \frac{h(x,y) \cdot \frac{\partial g}{\partial x} - g(x,y) \cdot \frac{\partial h}{\partial x}}{[h(x,y)]^2}$$
$$\frac{\partial z}{\partial y} = \frac{h(x,y) \cdot \frac{\partial g}{\partial y} - g(x,y) \cdot \frac{\partial h}{\partial y}}{[h(x,y)]^2}$$

Chain Rule: given $z = [g(x,y)]^n$

$$\frac{\partial z}{\partial x} = n\,[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial x}$$
$$\frac{\partial z}{\partial y} = n\,[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial y}$$

Further Examples:

For the function $U = U(x,y)$ find the the partial derivates with respect to x and y for each of the following examples

Example 1

$$U = -5x^3 - 12xy - 6y^5$$

Answer:

$$\frac{\partial U}{\partial x} = U_x = 15x^2 - 12y$$
$$\frac{\partial U}{\partial y} = U_y = -12x - 30y^4$$

Example 2

$$U = 7x^2y^3$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 14xy^3\\ \frac{\partial U}{\partial y} &= U_y = 21x^2y^2\end{aligned}$$

Example 3

$$U = 3x^2(8x - 7y)$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 3x^2(8) + (8x - 7y)(6x) = 72x^2 - 42xy\\ \frac{\partial U}{\partial y} &= U_y = 3x^2(-7) + (8x - 7y)(0) = -21x^2\end{aligned}$$

Example 4

$$U = (5x^2 + 7y)(2x - 4y^3)$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = (5x^2 + 7y)(2) + (2x - 4y^3)(10x)\\ \frac{\partial U}{\partial y} &= U_y = (5x^2 + 7y)(-12y^2) + (2x - 4y^3)(7)\end{aligned}$$

Example 5

$$U = \frac{9y^3}{x - y}$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = \frac{(x - y)(0) - 9y^3(1)}{(x - y)^2} = \frac{-9y^3}{(x - y)^2}\\ \frac{\partial U}{\partial y} &= U_y = \frac{(x - y)(27y^2) - 9y^3(-1)}{(x - y)^2} = \frac{27xy^2 - 18y^3}{(x - y)^2}\end{aligned}$$

Example 6

$$U = (x - 3y)^3$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 3(x - 3y)^2(1) = 3(x - 3y)^2\\ \frac{\partial U}{\partial y} &= U_y = 3(x - 3y)^2(-3) = -9(x - 3y)^2\end{aligned}$$

A Special Function: Cobb-Douglas

The Cobb-douglas function is a mathematical function that is very popular in economic models. The general form is

$$z = x^a y^b$$

and its partial derivatives are

$\partial z/\partial x = a x^{a-1} y^b$ and $\partial z/\partial y = b x^a y^{b-1}$

Furthermore, the slope of the level curve of a Cobb-douglas is given by

$$\frac{\partial z/\partial x}{\partial z/\partial y} = MRS = \frac{a}{b}\frac{y}{x}$$

Differentials

Given the function

$$y = f(x)$$

the derivative is

$$\frac{dy}{dx} = f'(x)$$

However, we can treat dy/dx as a fraction and factor out the dx

$$dy = f'(x)dx$$

where dy and dx are called *differentials*. If dy/dx can be interpreted as "the slope of a function", then dy is the "rise" and dx is the "run". Another way of looking at it is as follows:

- $dy =$ the change in y
- $dx =$ the change in x
- $f'(x) =$ how the change in x causes a change in y

Example 7 *if*

$$y = x^2$$

then

$$dy = 2xdx$$

Lets suppose $x = 2$ and $dx = 0.01$. What is the change in $y(dy)$?

$$dy = 2(2)(0.01) = 0.04$$

Therefore, at $x = 2$, if x is increased by 0.01 then y will increase by 0.04.

The two variable case

If

$$z = f(x, y)$$

then the change in z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad or \quad dz = f_x dx + f_y dy$$

which is read as "the change in z (dz) is due partially to a change in x (dx) plus partially due to a change in y (dy). For example, if

$$z = xy$$

then the total differential is

$$dz = ydx + xdy$$

and, if

$$z = x^2 y^3$$

then

$$dz = 2xy^3 dx + 3x^2 y^2 dy$$

REMEMBER: When you are taking the total differential, you are just taking all the partial derivatives and adding them up.

Example 8 *Find the total differential for the following utility functions*

1. $U(x_1, x_2) = ax_1 + bx_2 \qquad (a, b > 0)$
2. $U(x_1, x_2) = x_1^2 + x_2^3 + x_1 x_2$
3. $U(x_1, x_2) = x_1^a x_2^b$

Answers:

1. $$\begin{aligned} \frac{\partial U}{\partial x_1} &= U_1 = a \\ \frac{\partial U}{\partial x_2} &= U_2 = b \\ dU &= U_1 dx_1 + U_2 dx_2 = adx_1 + bdx_2 \end{aligned}$$

2. $$\begin{aligned} \frac{\partial U}{\partial x_1} &= U_1 = 2x_1 + x_2 \\ \frac{\partial U}{\partial x_2} &= U_2 = 3x_2^2 + x_1 \\ dU &= U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + (3x_2^2 + x_1)dx_2 \end{aligned}$$

3. $$\begin{aligned} \frac{\partial U}{\partial x_1} &= U_1 = ax_1^{a-1}x_2^b = \frac{ax_1^a x_2^b}{x_1} \\ \frac{\partial U}{\partial x_2} &= U_2 = bx_1^a x_2^{b-1} = \frac{bx_1^a x_2^b}{x_2} \\ dU &= \left(\frac{ax_1^a x_2^b}{x_1}\right) dx_1 + \left(\frac{bx_1^a x_2^b}{x_2}\right) dx_2 = \left[\frac{adx_1}{x_1} + \frac{bdx_2}{x_2}\right] x_1^a x_2^b \end{aligned}$$

The Implicit Function Theorem

Suppose you have a function of the form

$$F(y, x_1, x_2) = 0$$

where the partial derivatives are $\partial F/\partial x_1 = F_{x_1}$, $\partial F/\partial x_2 = F_{x_2}$and $\partial F/\partial y = F_y$. This class of functions are known as implicit functions where $F(y, x_1, x_2) = 0$ implicity define $y = y(x_1, x_2)$. What this means is that it is possible (theoretically) to rewrite to get y isolated and expressed as a function of x_1 and x_2. While it may not be possible to explicitly solve for y as a function of x, we can still find the effect on y from a change in x_1 or x_2 by applying the implicit function theorem:

Theorem 9 *If a function*

$$F(y, x_1, x_2) = 0$$

has well defined continuous partial derivatives

$$\begin{aligned} \frac{\partial F}{\partial y} &= F_y \\ \frac{\partial F}{\partial x_1} &= F_{x_1} \\ \frac{\partial F}{\partial x_2} &= F_{x_2} \end{aligned}$$

and if, at the values where F is being evaluated, the condition that

$$\frac{\partial F}{\partial y} = F_y \neq 0$$

holds, then y is implicitly defined as a function of x. The partial derivatives of y with respect to x_1 and x_2, are given by the ratio of the partial derivatives of F, or

$$\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y} \qquad i = 1, 2$$

To apply the implicit function theorem to find the partial derivative of y with respect to x_1 (for example), first take the total differential of F

$$dF = F_y dy + F_{x_1} dx_1 + F_{x_2} dx_2 = 0$$

then set all the differentials except the ones in question equal to zero (i.e. set $dx_2 = 0$) which leaves

$$F_y dy + F_{x_1} dx_1 = 0$$

or

$$F_y dy = -F_{x_1} dx_1$$

dividing both sides by F_y and dx_1 yields

$$\frac{dy}{dx_1} = -\frac{F_{x_1}}{F_y}$$

which is equal to $\frac{\partial y}{\partial x_1}$ from the implicit function theorem.

Example 10 *For each $f(x,y)=0$, find dy/dx for each of the following:*

1. $$y-6x+7=0$$

   Answer:
   $$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{(-6)}{1}=6$$

2. $$3y+12x+17=0$$

   Answer:
   $$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{(-12)}{3}=4$$

3. $$x^2+6x-13-y=0$$

   Answer:
   $$\frac{dy}{dx}=-\frac{f_x}{f_y}=\frac{-(2x+6)}{-1}=2x+6$$

4. $$f(x,y)=3x^2+2xy+4y^3$$

   Answer:
   $$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{6x+2y}{12y^2+2x}$$

5. $$f(x,y)=12x^5-2y$$

   Answer:
   $$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{60x^4}{-2}=30x^4$$

6. $$f(x,y)=7x^2+2xy^2+9y^4$$

   Answer:
   $$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{14x+2y^2}{36y^3+4xy}$$

Example 11 *For $f(x,y,z)$ use the implicit function theorem to find dy/dx and dy/dz :*

1. $$f(x,y,z)=x^2y^3+z^2+xyz$$

   Answer:
   $$\tfrac{dy}{dx}=-\tfrac{fx}{fy}=-\tfrac{2xy^3+yz}{3x^2y^2+xz}$$
   $$\tfrac{dy}{dz}=-\tfrac{fz}{fy}=-\tfrac{2z+xy}{3x^2y^2+xz}$$

2.

$$f(x,y,z) = x^3z^2 + y^3 + 4xyz$$

Answer:

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{3x^2z^2+4yz}{3y^2+4xz}$$

$$\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2x^3z+4xy}{3y^2+4xz}$$

3.

$$f(x,y,z) = 3x^2y^3 + xz^2y^2 + y^3zx^4 + y^2z$$

Answer:

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{6xy^3+z^2y^2+4y^3zx^3}{9x^2y^2+2xz^2y+3y^2zx^4+2yz}$$

$$\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2xzy^2+y^3x^4+y^2}{9x^2y^2+2xz^2y+3y^2zx^4+2yz}$$