OPMT 5701 Lecture Notes

1 Matrix Algebra

1. Gives us a shorthand way of writing a large system of equations.
2. Allows us to test for the existance of solutions to simultaneous systems.
3. Allows us to solve a simultaneous system.

DRAWBACK: Only works for linear systems. However, we can often covert non-linear to linear systems.
Example

$$y = ax^b$$
$$\ln y = \ln a + b \ln x$$

Matrices and Vectors
Given

$$\begin{aligned} y &= 10 - x \quad \Rightarrow \quad x + y = 10 \\ y &= 2 + 3x \quad \Rightarrow \quad -3x + y = 2 \end{aligned}$$

In matrix form

$$\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

Matrix of Coefficients   Vector of Unknows   Vector of Constants

In general

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = d_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = d_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = d_m \end{gathered}$$

n-unknowns $(x_1, x_2, \ldots x_n)$

Matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

Matrix shorthand

$$Ax = d$$

Where:
A= coefficient martrix or an array
x= vector of unknowns or an array
d= vector of constants or an array

Subscript notation

$$a_{ij}$$

is the coefficient found in the i-th row (i=1,. . . ,m) and the j-th column (j=1,. . . ,n)

1.1 Vectors as special matrices

The number of rows and the number of columns define the DIMENSION of a matrix.

A is m rows and n is columns or "mxn."

A matrix containing 1 column is called a "column VECTOR"

x is a n×1 column vector

d is a m×1 column vector

If x were arranged in a horizontal array we would have a row vector.

Row vectors are denoted by a prime

$$x' = [x_1, x_2, \ldots, x_n]$$

A 1×1 vector is known as a scalar.

$$x = [4] \text{ is a scalar}$$

Matrix Operators

If we have two matrices, A and B, then

$$A = B \quad iff \quad a_{ij} = b_{ij}$$

Addition and Subtraction of Matrices

Suppose A is an m×n matrix and B is a p×q matrix then A and B is possible only if m=p and n=q. Matrices must have the same dimensions.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Subtraction is identical to addition

$$\begin{bmatrix} 9 & 4 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} (9-7) & (4-2) \\ (3-1) & (1-6) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$$

Scalar Multiplication

Suppose we want to multiply a matrix by a scalar

$$\begin{matrix} k & \times & A \\ 1 \times 1 & & m \times n \end{matrix}$$

We multiply every element in A by the scalar k

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \ldots & ka_{1n} \\ ka_{21} & ka_{22} & \ldots & ka_{2n} \\ \vdots & & & \\ ka_{m1} & ka_{m2} & \ldots & ka_{mn} \end{bmatrix}$$

Example

Let k=[3] and A=$\begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$

then kA=

$$kA = \begin{bmatrix} 3\times 6 & 3\times 2 \\ 3\times 4 & 3\times 5 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 12 & 15 \end{bmatrix}$$

Multiplication of Matrices

To multiply two matrices, A and B, together it must be true that for

$$\begin{matrix} A & \times & B & = & C \\ m \times n & & n \times q & & m \times q \end{matrix}$$

That A must have the same number of columns (n) as B has rows (n).

The product matrix, C, will have the same number of rows as A and the same number of columns as B.

Example

$$\begin{array}{ccccc} A & \times & B & = & C \\ (1\times 3) & & (3\times 4) & & (1\times 4) \\ 1row & & 3rows & & 1row \\ 3cols & & 4cols & & 4cols \end{array}$$

In general

$$\begin{array}{ccccccccc} A & \times & B & \times & C & \times & D & = & E \\ (3\times 2) & & (2\times 5) & & (5\times 4) & & (4\times 1) & & (3\times 1) \end{array}$$

To multiply two matrices:
(1) Multiply each element in a given row by each element in a given column
(2) Sum up their products

Example 1

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Where:
c_{11}=$a_{11}b_{11} + a_{12}b_{21}$ (sum of row 1 times column 1)
c_{12}=$a_{11}b_{12} + a_{12}b_{22}$ (sum of row 1 times column 2)
c_{21}=$a_{21}b_{11} + a_{22}b_{21}$ (sum of row 2 times column 1)
c_{22}=$a_{21}b_{12} + a_{22}b_{22}$ (sum of row 2 times column 2)
Example 2

$$\begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (3\times 1) & +(2\times 3) & (3\times 2) & +(2\times 4) \end{bmatrix} = \begin{bmatrix} 9 & 14 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} (3\times 2) & +(2\times 1) & +(1\times 4) \end{bmatrix} = [12]$$

12 is the inner product of two vectors.

Suppose

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{then} \quad x' = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

therefore

$$x'x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1^2 + x_2^2]$$

However

$$xx' = 2 \text{ by } 2 \text{ matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix}$$

Example 4

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$Ab = \begin{bmatrix} (1\times 5) & + & (3\times 9) \\ (2\times 5) & + & (8\times 9) \\ (4\times 5) & + & (0\times 9) \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}$$

Example

$$Ax = d$$

$$\begin{array}{ccccc} A & & x & & d \\ \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} & = & \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \\ (3 \times 3) & & (3 \times 1) & & (3 \times 1) \end{array}$$

This produces

$$\begin{gathered} 6x_1 + 3x_2 + x_3 = 22 \\ x_1 + 4x_2 - 2x_3 = 12 \\ 4x_1 - x_2 + 5x_3 = 10 \end{gathered}$$

1.1.1 National Income Model

$$y = c + I_0 + G_0$$

$$C = a + bY$$

Arrange as

$$\begin{gathered} y - C = I_0 + G_0 \\ -bY + C = a \end{gathered}$$

Matrix form

$$\begin{array}{cccc} A & x & & = d \\ \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} & \begin{bmatrix} Y \\ C \end{bmatrix} & = & \begin{bmatrix} I_o + G_o \\ a \end{bmatrix} \end{array}$$

1.1.2 Division in Matrix Algebra

In ordinary algebra

$$\frac{a}{b} = c$$

is well defined iff b$\neq$ 0.
Now $\frac{1}{b}$ can be rewritten as b^{-1}, therefore $ab^{-1} = c$, also b$^{-1}a = c$.

But in matrix algebra

$$\frac{A}{B} = C$$

is not defined. However,

$$AB^{-1} = C$$

is well defined. BUT

$$AB^{-1} \neq B^{-1}A$$

B^{-1} is called the inverse of B

$$B^{-1} \neq \frac{1}{B}$$

In some ways B^{-1} has the same properties as b^{-1} but in other ways it differs. We will explore these differences later.

1.2 Linear Dependance

Suppose we have two equations

$$x_1 + 2x_2 = 1$$
$$3x_1 + 6x_2 = 3$$

To solve

$$3\left[-2x_2 + 1\right] - 6x_2 = 3$$
$$6x_2 + 3 - 6x_2 = 3$$
$$3 = 3$$

There is no solution. These two equations are linearly dependent. Equation 2 is equal to two times equation one.

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$Ax = d$$

where A is a two column vectors

$$\begin{bmatrix} U_1 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} U_2 \\ 2 \\ 6 \end{bmatrix}$$

Or A is two row vector

$$V_1' = \quad [1 \quad 2]$$
$$V_2' = \quad [3 \quad 6]$$

Where column two is twice column one and/or row two is three times row one

$$2U_1 = U_2 \text{ or } 3V_1' = V_2'$$

Linear Dependence Generally:

A set of vectors is said to be linearly dependent iff any one of them can be expressed as a linear combination of the remaining vectors.

Example:

Three vectors,

$$V_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \quad V_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

are linearly dependent since

$$3V_1 - 2V_2 = V_3$$
$$\begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

or expressed as

$$3V_1 - 2V_2 - V_3 = 0$$

General Rule

A set of vectors, $\mathrm{V}_1, V_{2,...,} V_n$ are linearly dependent if there exsists a set of scalars (i=1,...,n). Not all equal to zero, such that

$$\sum_{i=1}^{n} = k_i V_i = 0$$

Note

$$\sum_{i=1}^{n} k_i V_i = k_1 V_1 + k_2 V_2 + \ldots + k_n V_n$$

1.3 Commutative, Associative, and Distributive Laws

From Highschool algebra we know commutative law of addition,

$$a + b = b + a$$

commutative law of multiplication,

$$ab = ba$$

Associative law of addition,

$$(a + b) + c = a + (b + c)$$

associative law of multiplication,

$$(ab)c = a(bc)$$

Distributive law

$$a(b + c) = ab + ac$$

In matrix algebra most, but not all, of these laws are true.

1.3.1 I) Communicative Law of Addition

$$A + B = B + A$$

Since we are adding individual elements and $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for all i and j.

1.3.2 II) Similarly Associative Law of Addition

$$A + (B + C) = (A + B) + C$$

for the same reasons.

1.3.3 III) Matrix Multiplication

Matrix multiplication in not communtative

$$IB \neq BA$$

Example 1
Let A be 2×3 and B be 3×2

$$\begin{array}{ccccccccccc} A & \times & B & = & C & whereas & B & \times & A & = & C \\ (2 \times 3) & & (3 \times 2) & & (2 \times 2) & & (3 \times 2) & & (2 \times 3) & & (3 \times 3) \end{array}$$

Example 2
Let A=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$

$$AB = \begin{bmatrix} (1 \times 10) + (2 \times 6) & (1 \times -1) + (2 \times 7) \\ (3 \times 0) + (4 \times 6) & (3 \times -1) + (4 \times 7) \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

But

$$BA = \begin{bmatrix} (0)(1) - (1)(3) & (0)(2) - (1)(4) \\ (6)(1) + (7)(3) & (6)(2) + (7)(4) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

Therefore, we realize the distinction of post multiply and pre multiply. In the case

$$AB = C$$

B is pre multiplied by A, A is post multiplied by B.

1.3.4 IV) Associative Law

Matrix multiplication is associative

$$(AB)C = A(BC) = ABC$$

as long as their dimensions conform to our earlier rules of multiplication.

$$\begin{array}{ccccc} A & \times & B & \times & C \\ (m \times n) & & (n \times p) & & (p \times q) \end{array}$$

1.3.5 V) Distributive Law

Matrix multiplication is distributive

$$\begin{array}{ll} A(B+C) = AB + AC & \text{Pre multiplication} \\ (B+C)A = BA + CA & \text{Post multiplication} \end{array}$$

1.4 Identity Matrices and Null Matrices

1.4.1 Identity matrix:

is a square matrix with ones on its principal diagonals and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_n = \begin{bmatrix} 1 & 0 & \dots & n \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

Identity Matrix in scalar algebra we know

$$1 \times a = a \times 1 = a$$

In matrix algebra the identity matrix plays the same role

$$IA = AI = A$$

Example 1

Let A= $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} (1 \times 1) + (0 \times 2) & (1 \times 3) + (0 \times 4) \\ (0 \times 1) + (1 \times 2) & (0 \times 3) + (1 \times 4) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Example 2

Let A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A\,\{I_2 Case\}$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A\,\{I_3 Case\}$$

Furthermore,

$$\begin{array}{ccccccc} AIB & = & (AI)B & = & A(IB) & = & AB \\ (m \times n)(n \times p) & & & & & & (m \times n)(n \times p) \end{array}$$

1.4.2 Null Matrices

A null matrix is simply a matrix where all elements equal zero.

$$\underset{(2\times 2)}{0=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \quad \underset{(2\times 3)}{0=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

The rules of scalar algebra apply to matrix algebra in this case.

Example

$$a+0=a \Rightarrow \{scalar\}$$

$$A+0=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=A \qquad \{matrix\}$$

$$A\times 0=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}=0$$

1.5 Idiosyncracies of matrix algebra

1) We know AB≠BA

2)ab=0 implies a or b=0

In matrix

$$AB=\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

1.5.1 Transposes and Inverses

1)Transpose: is when the rows and columns are interchanged.

Transpose of A=A'or A^T

Example

If A=$\begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$

A'=$\begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix}$ and B'=$\begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$

Symmetrix Matrix

If A=$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$ then A'=$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$

A is a symmetric matrix.

Properties of Transposes

1) (A')'=A

2) (A+B)'=A'+B'

3) (AB)'=B'A'

Inverses and their Properties

In scalar algebra if

$$ax=b$$

then

$$x = \frac{b}{a} \text{ or } ba^{-1}$$

In matrix algebra
if

$$Ax = d$$

then

$$x = A^{-1}d$$

where A^{-1} is the inverse of A.

Properties of Inverses
1) Not all matrices have inverses
   non-singular: if there is an inverse
   singular: if there is no inverse
2) A matrix must be square in order to have an inverse. (Necessary but not sifficient)
3) In scalar algebra $\frac{a}{a} = 1$, in matrix algebra $AA^{-1} = A^{-1}A = I$
4) If an inverse exists then it must be unique.

Example
Let A=$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2} \end{bmatrix}$

$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ by factoring $\{\frac{1}{6}$ is a scalar$\}$
Post Multiplication

$$AA^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Pre Multiplication

$$A^{-1}A = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Further properties
If A and B are square and non-singular then:
1) $(A^{-1})^{-1} = A$
2) $(AB)^{-1} = B^{-1}A^{-1}$
3) $(A')^{-1} = (A^{-1})^{1}$

Solving a linear system

Suppose

$$\underset{(3\times 3)}{A} \quad \underset{(3\times 1)}{x} \quad = \quad \underset{(3\times 1)}{d}$$

then

$$\underset{(3\times 3)}{A^{-1}} \quad \underset{(3\times 3)}{A} \quad \underset{(3\times 1)}{x} \quad = \quad \underset{(3\times 3)}{A^{-1}} \quad \underset{(3\times 1)}{d}$$

$$\underset{(3\times 3)}{I} \quad \underset{(3\times 1)}{x} \quad = \quad \underset{(3\times 3)}{A^{-1}} \quad \underset{(3\times 1)}{d}$$

$$x = A^{-1}d$$

Example

$$Ax = d$$

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \quad A^{-1} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}$$

then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}\begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$x_1^* = 2 \quad x_2^* = 3 \quad x_3^* = 1$$

1.6 Linear Dependence and Determinants

Suppose we have the following

$$\begin{array}{ll} 1. & x_1 + 2x_2 = 1 \\ 2. & 2x_1 + 4x_2 = 2 \end{array}$$

where equation two is twice equation one. Therefore, there is no solution for x_1, x_2.

In matrix form:

$$Ax = d$$

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}\overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{d}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

The determinant of the coefficient matrix is

$$|A| = (1)(4) - (2)(2) = 0$$

a determinant of zero tells us that the equations are linearly dependent. Sometimes called a "vanishing determinant."

In general, the determinant of a square matrix, A is written as $|A|$ or detA.

For two by two case

$$|A| = \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} = a_{11}a_{22} - a_{12}a_{21} = k$$

where k is unique
any k$\neq$ 0 implies linear independence

Example 1
A=$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

$$|A| = (3 \times 5) - (1 \times 2) = 13 \qquad \{\text{Non-singular}\}$$

Example 2
B=$\begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$

$$|B| = (2 \times 24) - (6 \times 8) = 0 \qquad \{\text{Singular}\}$$

Three by three case
Given A=$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then

$$|A| = (a_1b_2c_3) + (a_2b_3c_1) + (b_1c_2a_3) - (a_3b_2c_1) - (a_2b_1c_3) - (b_3c_2a_1)$$

Cross-diagonals

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Use viso to display cross diagonals

Multiple along the diagonals and add up their products
⇒The product along the BLUE lines are given a positive sign
⇒The product of the RED lines are negative.

1.7 Using Laplace expansion

⇒ The cross diagonal method does not work for matrices greater than three by three
⇒ Laplace expansion evaluates the determinant of a matrix, A, by means of subdeterminants of A.

Subdeterminants or Minors

Given A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

By deleting the first row and first column, we get

$$|M_{11}| = \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}$$

The determinant of this matrix is the minor element a_1.
$|M_{ij}| \equiv$ is the subdeterminant from deleting the i-th row and the j-th column.

Given A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then

$$M_{21} \equiv \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} \quad M_{31} \equiv \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

1.7.1 Cofactors

A cofactor is a minor with a specific algebraic sign.

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

therefore

$$C_{11} = (-1)^2 |M_{11}| = |M_{11}|$$

$$C_{21} = (-1)^3 |M_{21}| = - |M_{21}|$$

The determinant by Laplace
Expanding down the first column

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11} |C_{11}| + a_{21} |C_{21}| + a_{31} |C_{31}| = \sum_{i=1}^{3} a_{i1} |C_{i1}|$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Note: minus sign (-1)$^{(1+2)}$

$$|A| = a_{11}\left[a_{22}a_{33} - a_{23}a_{32}\right] - a_{21}\left[a_{12}a_{33} - a_{13}a_{32}\right] + a_{31}\left[a_{12}a_{23} - a_{13}a_{22}\right]$$

Laplace expansion can be used to expand along any row or any column.

Example
Third row

$$|A| = a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example

A= $\begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

(1)Expand the first column

$$|A| = 8\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$|A| = (8 \times 0) - (4 \times 3) + (6 \times 1) = -6$$

(2)Expand the second column

$$|A| = -1\begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + 0\begin{vmatrix} 8 & 3 \\ 6 & 3 \end{vmatrix} - 0\begin{vmatrix} 8 & 3 \\ 4 & 1 \end{vmatrix}$$

$$|A| = (-1 \times 6) + (0) - (0) = -6$$

Suggestion: Try to choose an easy row or column to expand. (i.e. the ones with zero's in it.)

1.8 Rank of a Matrix

Definition

The rank of a matrix is the maximum number linearly independent rows in the matrix.

If A is an m×n matrix, then the rank of A is

$$r(A) \leq \min\left[m, n\right]$$

Read as: the rank of A is less than or equal to the minimum of m or n.

Using Determinants to Find the Rank

(1) If A is n×m and $|A|$=0

(2) Then delete one row and one column, and find the determinant of this new (n-1)×(n-1) matrix.

(3) Continue this process until you have a non-zero determinant.

1.9 Matrix Inversion

Given an n×n matrix, A, the inverse of A is

$$A^{-1} = \frac{1}{|A|} \bullet AdjA$$

where AdjA is the adjoint matrix of A. AdjA is the transpose of matrix A's cofactor matrix. It is also the adjoint, which is an n×n matrix

Cofactor Matrix (denoted C)
The cofactor matrix of A is a matrix who's elements are the cofactors of the elements of A

$$If\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

Example
Let A=$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \Rightarrow |A|$=-2
Step 1: Find the cofactor matrix

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

Step 2: Transpose the cofactor matrix

$$C^T = AdjA = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

Step 3: Multiply all the elements of AdjA by $\frac{1}{|A|}$ to find A^{-1}

$$A^{-1} = \frac{1}{|A|} \bullet AdjA = \left(-\frac{1}{2}\right) \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Step 4: Check by AA$^{-1} = I$

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} (3)(0) + (2)(\frac{1}{2}) & (3)(1) + (2)(-\frac{3}{2}) \\ (1)(0) + (0)(\frac{1}{2}) & (1)(1) + (0)(-\frac{3}{2}) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.10 Cramer's Rule

Suppose:

$$\text{Equation 1} \quad a_1x_1 + a_2x_2 = d_1$$

$$\text{Equation 2} \quad b_1x_1 + b_2x_2 = d_2$$

or

$$\begin{array}{cccc} A & x & = & d \\ \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \end{array}$$

where

$$A = a_1b_2 - a_2b_1 \neq 0$$

Solve for x_1 by substitution
From equation 1

$$x_2 = \frac{d_1 - a_1x_1}{a_2}$$

and equation 2

$$x_2 = \frac{d_2 - b_1 x_1}{b_2}$$

therefore:

$$\frac{d_1 - a_1 x_1}{a_2} = \frac{d_2 - b_1 x_1}{b_2}$$

Cross multiply

$$d_1 b_2 - a_1 b_2 x_1 = d_2 a_2 - b_1 a_2 x_1$$

Collect terms

$$d_1 b_2 - d_2 a_2 = (a_1 b_2 - b_1 a_2) x_1$$

$$x_1 = \frac{d_1 b_2 - d_2 a_2}{a_1 b_2 - b_1 a_2}$$

The denominator is the determinant of $|A|$
The numerator is the same as the denominator except $d_1 d_2$ replaces $a_1 b_1$.

Cramer's Rule

$$x_1 = \frac{\begin{vmatrix} d_1 & a_2 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{d_1 b_2 - d_2 a_2}{a_1 b_2 - b_1 a_2}$$

Where the d vector replaces column 1 in the A matrix

To find x_2 replace column 2 with the d vector

$$x_2 = \frac{\begin{vmatrix} a_1 & d_1 \\ b_1 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{a_1 d_2 - d_1 b_1}{a_1 b_2 - b_1 a_2}$$

Generally: to find x_i,replace column i with vector d; find the determinant.
x_i= the ratio of two determinants
$x_i = \frac{|A_i|}{|A|}$

1.10.1 Example: The Market Model

$$\text{Equation 1} \quad Q^d = 10 - P \quad \text{Or} \quad Q + P = 10$$

$$\text{Equation 2} \quad Q^s = P - 2 \quad \text{Or} \quad -Q + P = 2$$

Matrix form

$$\begin{array}{cccc} A & x & = & d \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} Q \\ P \end{bmatrix} & = & \begin{bmatrix} 10 \\ 2 \end{bmatrix} \end{array}$$

$$|A| = (1)(1) - (-1)(1) = 2$$

Find Q^e

$$Q^e = \frac{\begin{vmatrix} 10 & 1 \\ 2 & 1 \end{vmatrix}}{2} = \frac{10 - 2}{2} = 4$$

Find P^e

$$P^e = \frac{\begin{vmatrix} 1 & 10 \\ -1 & 2 \end{vmatrix}}{2} = \frac{2-(-10)}{2} = 6$$

Substitute P and Q into either equation 1 or equation 2 to verify

$$Q^d = 10 - P$$
$$10 - 6 = 4$$

1.10.2 Example: National Income Model

$$Y = C + I_0 + G_0 \quad \text{Or} \quad Y - C = I_0 + G_0$$

$$C = a + bY \quad \text{Or} \quad -bY + c = a$$

In matrix form $\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$

Solve for Y^e

$$Y^e = \frac{\begin{vmatrix} I_0 + G_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + G_0 + a}{1-b}$$

Solve for C^e

$$C^e = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1-b}$$