OPMT 5701 Lecture Notes
One Variable Optimization

October 10, 2006

Critical Points

A critical point occurs whenever the first derivative of a function is equal to zero, i.e. if

$$y = f(x)$$

then

$$\frac{dy}{dx} = f'(x) = 0$$

is a critical point. A critical is a stationary value of the function. A critical point can be

a. some kind of maximum point,

b. some kind of minimum point,

c. an inflection point.

Both a maximum and a minimum point can be a *relative* or *global* max or min (extremum). An inflection point is neither a maximum or a minimum. graphically, an inflection point is like a "shelf". All types of critical points are illustrated in figure one.

The First Derivative Test for relative Extremum

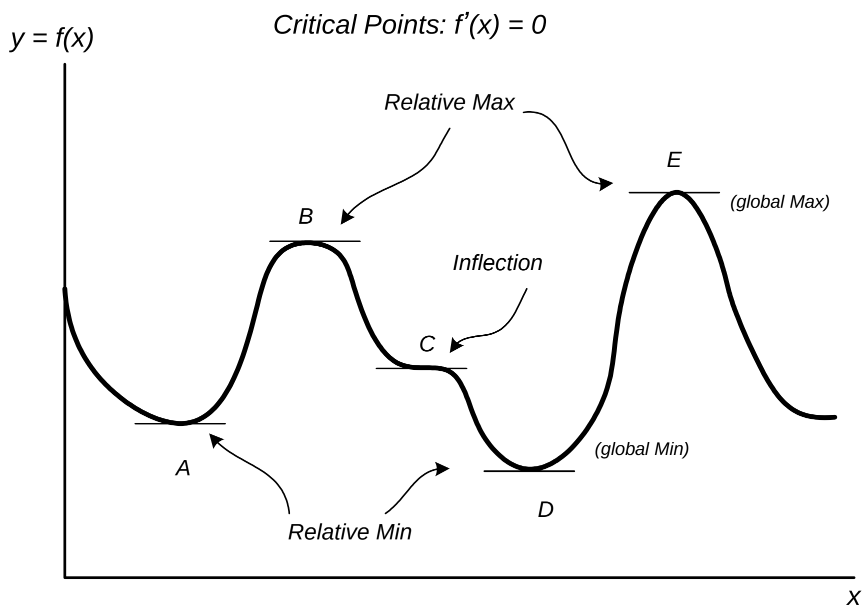
The condition

$$\frac{dy}{dx} = f'(x) = 0$$

is a necessary, but not sufficient, condition to establish an extremum (max or min). In order to establish whether a critical point is an extremum we can use the following test:
If, at $x = x_0$, $f'(x) = 0$, then $f(x_0)$ will be

(a) a relative Maximum if

$$\begin{array}{ll} f'(x) > 0 \quad for & x < x_0 \\ f'(x) < 0 \quad for & x > x_0 \end{array}$$



(b) a relative Minimum if

$$f'(x) < 0 \quad for \quad x < x_0$$
$$f'(x) > 0 \quad for \quad x > x_0$$

(c) an Inflection point if $f'(x)$ has the SAME SIGN for $x < x_0$ and $x > x_0$

Example: Suppose a revenue function is given by

$$R = 10x - x^2$$

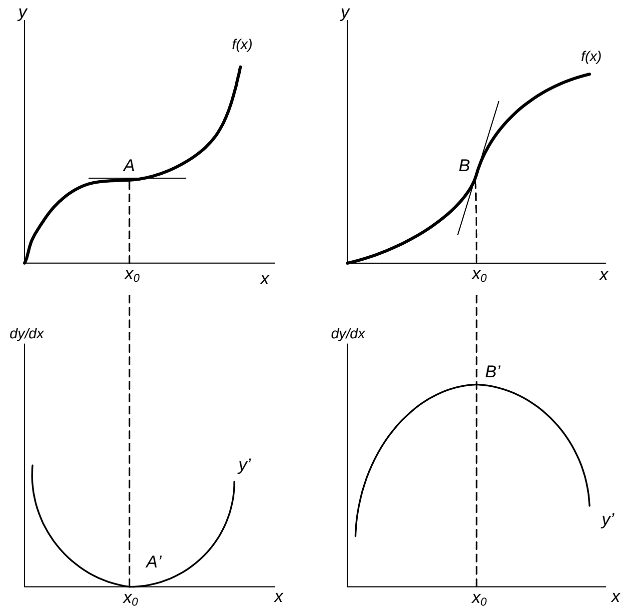
The first order condition is

$$\begin{aligned} R' &= 10 - 2x = 0 \\ x &= 5 \end{aligned}$$

Applying the first derivative test, we see that, for $x < 5$, $10 - 2x > 0$ and, for $x > 5$, $10 - 2x < 0$. Therefore we have a maximum.

More on Inflection Points

While $f'(x) = 0$ may be an inflection point, not all inflection points occur where the first derivative is zero. Figure two illustrates two types of inflection points. Also, each function's derivative is graphed directly below. Note the shape of each of the derivative functions: the first is *U-shaped* and the second is *dome-shaped*. Where the derivative function is at a max or a min is where the inflection occurs. Study the way the slope of the $f(x)$ function *behaves* on either side of the inflection (top graphs) and compare that to the *slope* of the derivative functions (bottom graphs).



Second and Higher Derivatives

Given that

$$y = f(x)$$

is a function, then

$$\frac{dy}{dx} = y' = f'(x)$$

is also a function. In theory, we can find the derivative of the function, $f'(x)$, which is

$$\frac{d(dy/dx)}{dx} = \frac{d^2y}{dx^2} = f''(x)$$

which is the SECOND derivative of the original function $f(x)$. If the second derivative exists, and is a well-defined function, we say that the original function $y = f(x)$ is a *continuous, twice differentiable* function.
Similarly, we can continue to take higher order derivatives (if they exist)

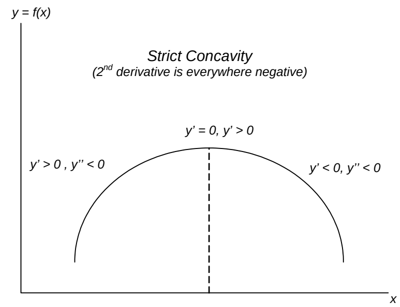
$$\begin{array}{cc} f'''(x) & 3rd \\ f^{(4)}(x) & 4th \\ \vdots & \vdots \\ f^{(n)}(x) & nth \end{array}$$

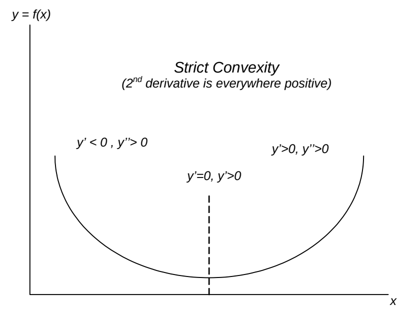
Interpretation of the Second Derivative

For the function

$$y = f(x)$$

the first derivative, $f'(x)$ measures the slope or rate of change of the function.





The second derivative, $f''(x)$ measures the rate of change *of the slope.* For example, if

$$f'(x) > 0$$

then $f(x)$ is upward (positive) sloped. If $f''(x) > 0$ then the function is getting "steeper". If, however, $f''(x) < 0$ the function is upward sloped but getting "flatter".
Illustration: If Distance $= f(t)$ where t is time traveled, then $f'(t) =$ velocity and $f''(x) =$ acceleration.

Concavity and Convexity

A function whose second derivative is always negative ($f''(x) < 0$) is referred to as a *strictly concave* function. An example is illustrated in figure 3 (top). Note that the first derivative can be positive ($f' > 0$), negative ($f' < 0$) or even a critical point ($f'' = 0$) but the second derivative is always negative. Using our example from the first derivative test,

$$R = 10x - x^2$$

where

$$\begin{aligned} R' &= 10 - 2x \\ R'' &= -2 \end{aligned}$$

Note that $R' \lesseqgtr 0$ depending if $x \gtreqless 5$ but $R'' = -2 < 0$ regardless of of the value of x.
Similarly, a function whose second derivative is everywhere positive is referred to a *strictly convex*[1]. An example of a strictly convex function is found in figure 3 (bottom)

[1] A function where $f''(x) \leq 0$ is said to be weakly concave (think of a dome with a *flat spot*). Similarly, if $f''(x) \geq 0$, then the function is weakly convex.

The Second Derivative Test

Another, often more reliable, test for a relative extremum is the second derivative test, which is the following:
Test:
Given the function $y = f(x)$ where, at $x = x_0$ $f'(x_0) = 0$, then $f(x_0)$ will be

- a relative maximum if $f''(x_0) < 0$
- a relative minimum if $f''(x_0) > 0$
- either a maximum, minimum or inflection point if $f''(x_0) = 0$. Essentially there is not enough information to determine the nature of the critical point. In this case, an additional test is required: either the first derivative test or the N^{th} derivative test (below).

Example: Find the critical points for

$$y = 24x - 3x^2 + 5$$

and determine if they are a maximum or minimum. first

$$\begin{aligned} y' &= 24 - 6x = 0 \\ x &= 4 \end{aligned}$$

check second order conditions

$$y'' = -6 < 0$$

therefore we have a maximum.

Example: Find the critical points for

$$y = \frac{32}{x} + 2x - 12$$

and determine if they are a maximum or minimum. first

$$\begin{aligned} y' &= -\frac{32}{x^2} + 2 = 0 \\ x^2 &= \frac{32}{2} = 16 \\ x &= 4, -4 \end{aligned}$$

therefore this function has two critical points. Now check second order conditions

$$y'' = (-2)(-32)x^{-3} = \frac{64}{x^3}$$

substituting in the values of x, we get

$$\begin{array}{llll} y'' = \frac{64}{(4)^3} = 1 & when & x = 4 & \text{(min)} \\ y'' = \frac{64}{(-4)^3} = -1 & when & x = -4 & \text{(max)} \end{array}$$

The Nth Derivative Test

If the second derivative test fails to reveal the nature of a critical point, we can apply the nth derivative test which is as follows:

1. Once a critical point has been found ($f'(x_0) = 0$), and the second derivative also equals zero ($f''(x_0) = 0$), continue to take derivatives until the first non-zero derivative is found.

2. If the derivative number is an even number (i.e. the fourth derivative, sixth derivative, etc.), apply the rules used in the second derivative test:

$$\begin{array}{lll} f^{(N)} < 0 & Max & \\ & & \{N = 4, 6, 8, \ldots\} \\ f^{(N)} > 0 & Min & \end{array}$$

3. If the first non-zero derivative occurs at an odd numbered derivative $\{N = 3, 5, 7, \ldots\}$ then this is an *Inflection Point*.

Example 1 Consider

$$f(x) = x^4$$

has a critical point at $x = 0$

$$f'(x) = 4x^3 = 0$$

check second, and higher derivatives when $x = 0$

$$\begin{aligned} f''(0) &= 12x^2 = 0 \\ f'''(0) &= 24x = 0 \\ f^{(4)}(0) &= 24 > 0 \end{aligned} \tag{1}$$

Since the first non-zero derivative occurs at the 4th derivative (even number) and is positive, this is a minimum (see figure 4)

Example 2 Now consider the following

$$f(x) = x^5$$

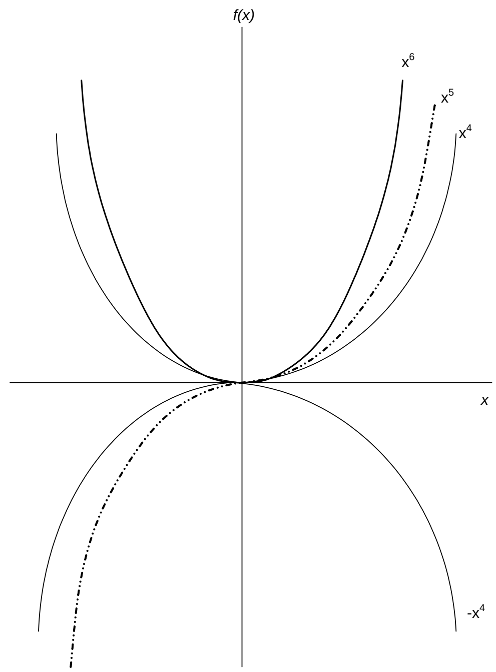
this function also has a critical point at $x = 0$

$$f'(0) = 5x^4 = 0$$

check the second and higher derivatives when $x = 0$

$$\begin{aligned} f''(0) &= 20x^3 = 0 \\ f'''(0) &= 60x^2 = 0 \\ f^{(4)}(0) &= 120x = 0 \\ f^{(5)}(0) &= 120 \neq 0 \end{aligned}$$

since the first non-zero derivative occurs at the 5th derivative (odd number) it is an inflection point (see figure 4)



Example 3 For the function

$$y = x^6$$

which has a critical point at $x = 0$, the derivatives are

$$\begin{aligned} f' &= 6x^5 \\ f'' &= 30x^4 \\ f''' &= 120x^3 \\ f^{(4)} &= 360x^2 \\ f^{(5)} &= 720x \\ f^{(6)} &= 720 \end{aligned}$$

. Since the sixth derivative is the first non=zero and is positive, this is also a minimum

Example 4 if

$$f(x) = -x^4$$

has a critical point at $x = 0$

$$f'(x) = -4x^3 = 0$$

check second, and higher derivatives when $x = 0$

$$\begin{aligned} f''(0) &= -12x^2 = 0 \\ f'''(0) &= -24x = 0 \\ f^{(4)}(0) &= -24 < 0 \end{aligned} \tag{2}$$

Since the first non-zero derivative occurs at the 4th derivative (even number) and is negative, this is a maximum (see figure 4)