OPMT 5701
Examples of Lagrange

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1 Lagrange Multiplier Approach

Create a new function called the Lagrangian:

$$L = f(x_1, x_2) + \lambda g(x_1, x_2)$$

since $g(x_1, x_2) = 0$ when the constraint is satisfied

$$L = f(x_1, x_2) + zero$$

We have created a new independent variable λ (lambda), which is called the Lagrangian Multiplier.

We now have a function of three variables; x_1, x_2, and λ

Now we Maximize

$$L = f(x_1, x_2) + \lambda g(x_1, x_2)$$

First Order Conditions

$$\begin{aligned} L_\lambda &= \frac{\partial L}{\partial \lambda} = g(x_1, x_2) = 0 \quad Eq.1 \\ L_1 &= \frac{\partial L}{\partial x_1} = f_1 + \lambda g_1 = 0 \quad Eq.2 \\ L_2 &= \frac{\partial L}{\partial x_2} = f_2 + \lambda g_2 = 0 \quad Eq.3 \end{aligned}$$

From Eq. 2 and 3 we get:

$$\frac{f_1}{f_2} = \frac{-\lambda g_1}{-\lambda g_2} = \frac{g_1}{g_2}$$

From the 3 F.O.C.'s we have 3 equations and 3 unknowns $(x_{1,}x_2, \lambda)$. In principle we can solve for x_1^*, x_2^*, and λ^*.

1.1 Example 1:

Let:

$$U = xy$$

Subject to:

$$10 = x + y \quad P_x = P_y = 1$$

Lagrange:

$$\begin{aligned} L &= f(x, y) + \lambda(g(x, y)) \\ L &= xy + \lambda(10 - x - y) \end{aligned}$$

F.O.C.

$$\begin{array}{cc} L_\lambda = 10 - x - y = 0 & Eq.1 \\ L_x = y - \lambda = 0 & Eq.2 \\ L_y = x - \lambda = 0 & Eq.3 \end{array}$$

From (2) and (3) we see that:

$$\tfrac{y}{x} = \tfrac{\lambda}{\lambda} = 1 \ \ \underline{\text{or}} \ \ y = x \qquad Eq.4$$

From (1) and (4) we get:

$$10 - x - x = 0 \ \ or \ \ x^* = 5 \ \ and \ \ y^* = 5$$

From either (2) or (3) we get:

$$\lambda^* = 5$$

1.2 Example 2: Utility Maximization

Maximize

$$u = 4x^2 + 3xy + 6y^2$$

subject to

$$x + y = 56$$

Set up the Lagrangian Equation:

$$L = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned} L_x &= 8x + 3y - \lambda = 0 \\ L_y &= 3x + 12y - \lambda = 0 \\ L_\lambda &= 56 - x - y = 0 \end{aligned}$$

From the first two equations we get

$$\begin{aligned} 8x + 3y &= 3x + 12y \\ x &= 1.8y \end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned} 56 - 1.8y - y &= 0 \\ y &= 20 \end{aligned}$$

therefore

$$x = 36 \qquad \lambda = 348$$

1.3 Example 3: Cost minimization

A firm produces two goods, x and y. Due to a government quota, the firm must produce subject to the constraint $x + y = 42$. The firm's cost functions is

$$c(x, y) = 8x^2 - xy + 12y^2$$

The Lagrangian is

$$L = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

The first order conditions are

$$\begin{aligned} L_x &= 16x - y - \lambda = 0 \\ L_y &= -x + 24y - \lambda = 0 \\ L_\lambda &= 42 - x - y = 0 \end{aligned} \tag{1}$$

Solving these three equations simultaneously yields

$$x = 25 \qquad y = 17 \qquad \lambda = 383$$

1.4 Example 4: Utility Max #2

Max:

$$U = x_1 x_2$$

Subject to:

$$B = P_1 x_1 + P_2 x_2$$

Langrange:

$$L = x_1 x_2 + \lambda \left(B - P_1 x_1 - P_2 x_2\right)$$

F.O.C.

$$\begin{array}{cr} L_\lambda = B - P_1 x_1 - P_2 x_2 = 0 & \text{Eq. 1} \\ L_1 = x_2 - \lambda P_1 = 0 & \text{Eq. 2} \\ L_2 = x_1 - \lambda P_2 = 0 & \text{Eq. 3} \end{array}$$

From Eq. (2) and (3) $\left(\frac{x_2}{x_1} = \frac{P_1}{P_2} = MRS\right)$

$$\begin{aligned} x_2 &= \lambda P_1 \\ x_1 &= \lambda P_2 \end{aligned}$$

divide top equation by the bottom

$$\frac{x_2}{x_1} = \frac{\lambda P_1}{\lambda P_2}$$

Cancel the λ from top/bottom of RHS

$$\frac{x_2}{x_1} = \frac{P_1}{P_2}$$

Solve for x_1^*
From (2) and (3)

$$x_2 = \frac{P_1}{P_2} x_1$$

Sub into (1) and simplify

$$\begin{aligned}
B &= P_1 x_1 + P_2 x_2 \\
B &= P_1 x_1 + P_2 \left(\frac{P_1}{P_2} x_1 \right) \\
B &= 2 P_1 x_1 \\
x_1^* &= \frac{B}{2P_1}
\end{aligned}$$

Substitute your answer for x_1^* into Eq 1

$$\begin{aligned}
B &= P_1x_1 + P_2x_2 \\
B &= P_1\left(\frac{B}{2P_1}\right) + P_2x_2 \\
B &= \frac{B}{2} + P_2x_2 \\
B - \frac{B}{2} &= P_2x_2 \\
\frac{B}{2} &= P_2x_2 \\
x_2^* &= \frac{B}{2P_2}
\end{aligned}$$

The solution to $x_1^*$and x_2^* are the Demand Functions for x_1 and x_2

1.5 Minimization and Lagrange

Min x, y

$$P_xx + P_yy$$

Subject to

$$U_0 = U(x, y)$$

Lagrange

$$L = P_xX + P_yY + \lambda(U_0 - U(x, y))$$

F.O.C.

$$\begin{aligned}
L_\lambda = U_0 - U(x, y) = 0 &\quad \text{Eq. 1} \\
L_x = P_x - \lambda\tfrac{\partial U}{\partial x} = 0 &\quad \text{Eq. 2} \\
L_y = P_y - \lambda\tfrac{\partial U}{\partial y} = 0 &\quad \text{Eq. 3}
\end{aligned}$$

From (2) and (3) we get

$$\underbrace{\frac{P_x}{P_y} = \frac{\lambda U_x}{\lambda U_y} = \frac{U_x}{U_y} = MRS}_{\text{(The same result as in the MAX problem)}}$$

Solving (1), (2), and (3), we get:

$$x^* = x(P_x, P_y, U_0) \qquad y^* = y(P_x, P_y, U_0) \qquad \lambda^* = \lambda(P_x, P_y, U_0)$$

1.5.1 Example (part 1)

Max

$$xy + \lambda(B - P_x x - P_y y)$$

F.O.C.'s

$$\underbrace{\begin{gathered} L_x = y - \lambda P_x = 0 \\ L_y = x - \lambda P_y = 0 \\ L_\lambda = B - P_x x - P_y y = 0 \end{gathered}}$$

$$x^* = \tfrac{B}{2P_x} \qquad y^* = \tfrac{B}{2P_y} \qquad \lambda^* = \tfrac{B}{2P_x P_y}$$

1.5.2 Example (part 2)

Min

$$P_x x + P_y y + \lambda(U_0 - xy)$$

F.O.C.'s

$$L_x = P_x - \lambda y = 0 \qquad (1)$$
$$L_y = P_y - \lambda x = 0 \qquad (2)$$
$$L_\lambda = U_0 - xy = 0 \qquad (3)$$

First, use equations (1) and (2) to eliminate λ

$$\begin{aligned} P_x &= \lambda y \\ P_y &= \lambda x \end{aligned}$$

divide (1) by (2)

$$\begin{aligned} \frac{P_x}{P_y} &= \frac{\lambda y}{\lambda x} \\ \frac{P_x}{P_y} &= \frac{y}{x} \\ y &= \frac{P_x}{P_y} x \end{aligned}$$

Substitute into eq (3)

$$\begin{aligned} U_0 &= xy \\ U_0 &= x\left(\frac{P_x}{P_y} x\right) \\ U_0 &= \frac{P_x}{P_y} x^2 \\ x^2 &= \frac{P_y}{P_x} U_0 \\ x &= \sqrt{\frac{P_y}{P_x} U_0} = \frac{P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}}} \end{aligned}$$

Follow the same procedure to find

$$y^* = \frac{P_x^{\frac{1}{2}} U_0^{\frac{1}{2}}}{P_y^{\frac{1}{2}}} \qquad \lambda^* = \frac{U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}} P_y^{\frac{1}{2}}}$$